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# MODEL PROBLEM OF INSTANTANEOUS MOTION OF A THREE-PHASE CONTACT LINE

**V. V. Pukhnachev and I. B. Semenova**

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*Hydrodynamic problems of fluid flow with three-phase contact lines (for example, solid body–liquid–gas or solid body and two nonmixing liquids) are of special interest. Much attention has been paid lately to steady and quasisteady flows. Significantly unsteady problems of this kind have almost escaped consideration. In the present paper, we study a model problem of a significantly unsteady motion of a finite volume of an incompressible fluid with a three-phase contact line. The static contact angle is assumed to be right and the initial free surface of the liquid is assumed to be cylindrical. One of the planes instantaneously begins to move toward the other with a constant finite velocity. Flows with high Reynolds numbers and small capillary numbers are considered. Mass forces are ignored in the problem. The basic result is the construction of a formal asymptotic of the solution at small times.*

**1. Formulation of the Problem.** We consider a model problem of dramatically unsteady motion of a finite volume of a viscous incompressible fluid enclosed between two infinite solid planes. Initially, these planes are located at a distance $2a$ from each other. The static contact angle (i.e., the angle between the free and solid boundaries) is $\pi/2$ and the free surface of the fluid is assumed to be cylindrical (with a circular cross section of radius $b$). One of the planes suddenly begins its motion toward the other with a constant velocity $V$. Flows with high Reynolds numbers Re and low capillary numbers Ca are considered (for example, for water at room temperature and the parameters $a = b = 10$ cm and $V = 10$ cm/sec, we have $\mathrm{Re} = Va/\nu = 10^4$ and $\mathrm{Ca} = \rho\nu V/\sigma = 1.33 \cdot 10^{-3}$, where $\nu$ is the kinematic viscosity of the fluid, $\sigma$ is the surface tension coefficient, and $\rho$ is the fluid density). The choice of a cylindrical free surface allows us to use in a specially marked central flow region the solution derived by L. V. Ovsyannikov [1] for a similar problem in the case of an ideal incompressible fluid. (A wide range of exact solutions of the Euler equations found by L. V. Ovsyannikov was studied in [2, 3].) Pukhnachev noted [4] that the corresponding solution obtained by Ovsyannikov also satisfies the conditions at the free boundary in the case of a viscous incompressible fluid. This gives grounds to assume that an unsteady boundary layer near the free surface does not appear. A similar problem for an unsteady boundary layer near solid surfaces without free boundaries was solved by Blasius [5]. A solution similar to his solution can be used in this problem for regions immediately adjacent to the solid boundary and remote from the free boundary. Thus, the problem actually reduces to determination of motion in the region located in an immediate vicinity from both solid and free boundaries, i.e., in the vicinity of the three-phase contact line. The resultant flow is assumed to be axisymmetric. In addition, mass forces are assumed to be absent.

We introduce a cylindrical coordinate system $(r, \theta, z)$, where the $z$ axis is directed along the fluid cylinder centerline. With account of the above assumptions, the Navier–Stokes equations in the cylindrical

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 4, pp. 51–61, July–August, 1999. Original article submitted November 11, 1997.



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coordinate system have the form

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\Big(\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}\Big), \tag{1.1}$$

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Big(\frac{\partial^2 v_z}{\partial r^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\Big), \tag{1.2}$$

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0, \tag{1.3}$$

where $v_r$ is the velocity-vector component in the direction of variation of the radius $r$, $v_z$ is the velocity-vector component along the $z$ axis, $p$ is the pressure in the fluid, and $t$ is the time.

The continuity equation (1.3) allows us to introduce the Stokes stream function $\psi$ so that $v_r = \partial\psi/(r\partial z)$ and $v_z = -\partial\psi/(r\partial r)$. Eliminating the pressure from Eqs. (1.1) and (1.2) by means of cross differentiation, we obtain

$$\frac{\partial\tilde{\Delta}\psi}{\partial t} = \nu\tilde{\Delta}\tilde{\Delta}\psi + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\tilde{\Delta}\psi}{\partial z} - \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\tilde{\Delta}\psi}{\partial r} + \frac{2}{r^2}\tilde{\Delta}\psi\frac{\partial\psi}{\partial r}, \tag{1.4}$$

where $\tilde{\Delta} = \partial^2/\partial r^2 - \partial/(r\partial r) + \partial^2/\partial z^2$ is the Stokes operator. This equation is considered now in an immediate vicinity of the solid boundary, but at a distance from the free boundary (so that the effect of the latter could be ignored for a while). In accordance with [5], in the case of motion started from a quiescent state, the term $\nu\tilde{\Delta}\tilde{\Delta}\psi$ is dominating at the initial time when the boundary layer is very thin yet (the boundary layer thickness is $\delta \sim \sqrt{\nu t}$), whereas the contribution of convective terms to the magnitude of acceleration is small. The term mentioned is balanced by the unsteady local acceleration $\partial\tilde{\Delta}\psi/\partial t$ and Eq. (1.4) can be asymptotically simplified:

$$\frac{\partial\tilde{\Delta}\psi}{\partial t} = \nu\tilde{\Delta}\tilde{\Delta}\psi. \tag{1.5}$$

In this region, we can seek an approximate solution by analogy with the asymptotic solution derived by Blasius [5] in the form $\psi_m = \sqrt{\nu t}r^2 cf(z/\sqrt{\nu t}) + O(Vt/a)$. Substitution of this representation of the stream function into (1.5) yields the following fourth-order ordinary differential equation for the function $f(\zeta)$:

$$-\frac{1}{2}f^{\rm II} - \frac{\zeta}{2}f^{\rm III} = f^{\rm IV} \Leftrightarrow -\Big(\frac{\zeta}{2}f^{\rm II}\Big)^{\rm I} = f^{\rm IV} \quad (\zeta = z/\sqrt{\nu t}).$$

Its integration gives $f^{\rm III} + (\zeta/2)f^{\rm II} = c_1$, where $c_1$ is a certain constant. Nevertheless, since the adhesion conditions should be fulfilled at the wall, from which it follows that $f(0) = f^{\rm I}(0) = 0$, and, in addition, the condition at infinity $f^{\rm I} \to 1$ should be fulfilled as $\zeta \to \infty$, we have $c_1 = 0$ and the above equation has the unique solution

$$f(\zeta) = \frac{1}{2\sqrt{\pi}}\int_0^\zeta dl\int_0^l \exp\Big(-\frac{m^2}{4}\Big)\,dm.$$

In this representation, the constant $c$ remains undetermined. Considering the flow in the central region of the cylinder, i.e., in the region that is not adjacent to the solid boundaries, we are justified in using the solution obtained by Ovsyannikov [1] for a similar problem with an ideal fluid. (The main difference between these two problems is observed in the vicinity of the solid wall with no-slip conditions for an ideal fluid and adhesion conditions for a viscous fluid.) In accordance with the solution of this equation, if the critical point is in the center of symmetry, the velocity field has the form

$$v_r = \frac{Vr}{2a(1 - Vt/a)}, \qquad v_z = -\frac{Vz}{a(1 - Vt/a)}.$$

If we introduce a new coordinate $z' = z + a(1 - Vt/a)$ (the radius remains unchanged: $r' = r$), the lower plane corresponds to $z' = 0$. Then, in the new coordinates fitted with the lower plane, we have $v_r = v_{r'} =$

$Vr'/(2a(1-Vt/a))$ and $v_z = -Vz'/(a(1-Vt/a)) + V = v_{z'} + V$ and, for $t = 0$, we have $v_{r'} = Vr'/(2a)$ and $v_{z'} = -Vz'/a$, from which we obtain the stream function $\psi = V(r')^2 z'/(2a)$. To relate the solutions of Ovsyannikov and Blasius, we choose now a previously undetermined constant $c = V/(2a)$. Then, we have

$$\psi_m = \sqrt{\nu t}\frac{Vr^2}{2a}\frac{1}{2\sqrt{\pi}}\int\limits_0^\zeta dl \int\limits_0^l \exp\Big(-\frac{m^2}{4}\Big)dm, \tag{1.6}$$

where $\psi_m$ is the main term in the expansion of the stream function of the main flow for $t \to 0$.

The solution $\psi_m$ found is invalid in the small vicinity of the three-phase contact line for two reasons. First, the solution obtained by analogy with the Blasius solution, as mentioned above, does not agree with the conditions on the free surface. Second, the attempt to continue the solution to the moving contact line with adhesion conditions preserved leads to the divergence of the Dirichlet integral [7, 8]. This means that the dissipation rate of the kinetic energy in the fluid is infinite and the solution has no physical sense. Thus, we should separately consider the small vicinity of the three-phase contact line and seek the corresponding solution there.

For this purpose, we consider a vicinity of the contact line of thickness $\delta = \sqrt{\nu t}$, which has the order of the boundary-layer thickness. We introduce the local coordinates $y = z$ and $x = r - b(1 - Vt/a)^{-1/2}$. The self-similarity of the main term in the asymptotic expansion (1.6) leads to the natural hypothesis of local self-similarity of the free surface and the flow in the chosen region. In turn, it follows from this hypothesis that the free-surface equation in terms of $(x, y)$ has the following form in the first approximation:

$$F(\xi, \eta) = 0. \tag{1.7}$$

Here $\xi = x/\sqrt{\nu t}$ and $\eta = y/\sqrt{\nu t}$. Substitution of (1.7) into the kinematic condition on the free surface [6] yields

$$-\frac{1}{2t}\Big(\xi\frac{\partial F}{\partial \xi} + \eta\frac{\partial F}{\partial \eta}\Big) + \frac{1}{\sqrt{t}}\Big(v_x\frac{\partial F}{\partial \xi} + v_y\frac{\partial F}{\partial \eta}\Big). \tag{1.8}$$

We assume that the velocity components $v_x = v_r - Vb/(2a)(1-Vt/a)^{-3/2}$ and $v_y = v_z - V$ are limited near the contact line. In this case, for small values of $t$, the second term in (1.8) can be ignored. Then the function $F$ depends only on $\theta_p = \arctan(\eta/\xi)$. It follows from (1.7) that $\theta_p = \text{const}$ is the free-surface equation. However, in the limit as $r_p = \sqrt{\xi^2 + \eta^2} \to \infty$, the meridional section of the free surface becomes a vertical straight line. Thus, in the polar coordinates $r_p$ and $\theta_p$, equality (1.7) yields $\theta_p = \pi/2$. This means that, for small $t$, the free boundary remains close to the cylindrical surface of radius $R(t) = b(1 - Vt/a)^{-1/2}$, including the contact region itself.

Obviously, in the first approximation in the considered vicinity of the contact line whose width is of the order of $\sqrt{\nu t}$, we can ignore the curvature of the line itself and consider the flow to be planar there. In the Cartesian coordinates $(x, y)$, the Stokes equation (1.5) is written as

$$\frac{\partial \Delta\tilde{\psi}}{\partial t} = \nu\Delta\Delta\tilde{\psi}, \tag{1.9}$$

where $\Delta$ is the Laplace operator and $\tilde{\psi}$ is the stream function determined so that $v_x = \partial\tilde{\psi}/\partial y$ and $v_y = -\partial\tilde{\psi}/\partial x$ are the corresponding components of the velocity.

Equation (1.9) has a partial solution $\tilde{\psi}$ independent of $x$: $\tilde{\psi}_p = \sqrt{\nu t}c_2 g(y/\sqrt{\nu t})$, which is important for agreement with the main term of the expansion of the stream function of the main flow. Substituting this expression into (1.9), we obtain the equation for the function $g(\eta)$: $g^{\mathrm{III}} + g^{\mathrm{II}}\eta/2 = 0$, which coincides with the equation for the function $f$ considered above. From the adhesion conditions on the solid boundary and from the conditions at infinity $g^{\mathrm{I}} \to 0$ and $\eta \to \infty$ we obtain

$$\tilde{\psi}_p = \sqrt{\nu t}\,\frac{Vb}{2a}\Big(\frac{1}{2\sqrt{\pi}}\int\limits_0^\eta dl \int\limits_0^l \exp\Big(-\frac{m^2}{4}\Big)\,dm - \eta\Big). \tag{1.10}$$

The constant $c_2 = Vb/(2a)$ is chosen from the condition of compatibility with the main flow.

At the next step, we seek the asymptotic solution

$$\tilde{\psi} = \sqrt{\nu t}\Phi(\xi, \eta) + O(t) \tag{1.11}$$

for $t \to 0$ of Eq. (1.9) written in the form

$$\Delta\Delta\Phi + \frac{1}{2}\Big(\xi\frac{\partial\Delta\Phi}{\partial\xi} + \eta\frac{\partial\Delta\Phi}{\partial\eta} + \Delta\Phi\Big) = 0. \tag{1.12}$$

In deriving the boundary conditions for $\Phi$, we use the fact that the boundary conditions on both the solid and free boundaries should be imposed for a "full" flow, which takes into account all the three components $\psi_m$, $\tilde{\psi}_p$, and $\tilde{\psi}$.

The kinematic equation on the free boundary [in the coordinate system $(x, y)$ or $(\xi, \eta)$ fitted with the free boundary] [6] is

$$(\tilde{\psi}_p + \tilde{\psi})_y + \tilde{v}_r = 0 \quad \text{for} \quad x = 0 \quad (\xi = 0),$$

where $\tilde{v}_r$ is the radial component of the main flow written in the coordinates $(x, y)$. In accordance with (1.6), (1.10), and (1.11), the latter condition is rewritten as

$$\sqrt{\nu t} g_\eta \frac{1}{\sqrt{\nu t}} + \sqrt{\nu t}\Phi_\eta \frac{1}{\sqrt{\nu t}} + \frac{Vb}{2a}\frac{1}{2\sqrt{\pi}}\int\limits_0^\eta \exp\Big(-\frac{m^2}{4}\Big)\,dm = 0.$$

Substituting here the expression for $g_\eta$

$$\Phi_\eta\Big|_{\xi=0} = \frac{Vb}{2a} - \frac{Vb}{2a}\frac{1}{\sqrt{\pi}}\int\limits_0^\eta \exp\Big(-\frac{m^2}{4}\Big)\,dm$$

and integrating this relation with respect to $\eta$, we obtain

$$\Phi\Big|_{\xi=0} = \frac{Vb}{2a}\Big(\eta - \frac{1}{\sqrt{\pi}}\int\limits_0^\eta dl \int\limits_0^l \exp\Big(-\frac{m^2}{4}\Big)\,dm\Big). \tag{1.13}$$

The general form of the dynamic condition at the free boundary is

$$(p_a - p)\boldsymbol{n} + 2\rho\nu D \cdot \boldsymbol{n} = 2\sigma H\boldsymbol{n}, \tag{1.14}$$

where $\boldsymbol{n}$ is the unit normal to the free surface, $D = (\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^*)/2$ is the strain-rate tensor, $H$ is the mean curvature of the surface, and $p_a$ is the atmospheric pressure. In our case, in accordance with (1.11), the normal stress $-p + 2\rho\nu\partial v_x/\partial x$ has the order of $1/\sqrt{t}$ for $t \to 0$; in accordance with (1.7), the capillary pressure $\sigma H$ is of the same order. Their ratio is proportional to the capillary number Ca, which is assumed to be small. Hence, the projection of the left side of Eq. (1.14) onto the normal to the free surface is $O(\mathrm{Ca})$. In turn, the shear stresses yield

$$\frac{\partial^2(\tilde{\psi} + \tilde{\psi}_p)}{\partial y^2} - \frac{\partial^2(\tilde{\psi} + \tilde{\psi}_p)}{\partial x^2} + \tilde{P}_{rz} = 0 \quad \text{for} \quad \xi = 0,$$

where $\tilde{P}_{rz}$ is the shear stress corresponding to the main flow. With account of formulas (1.6), (1.10), and (1.11), this equality can be rewritten as

$$\frac{1}{\sqrt{\nu t}}\Phi_{\xi\xi} + \frac{1}{\sqrt{\nu t}}(g_{\eta\eta} + \Phi_{\eta\eta}) + \frac{1}{\sqrt{\nu t}}\frac{Vb}{2a}\frac{1}{2\sqrt{\pi}}\exp\Big(-\frac{\eta^2}{4}\Big) = 0.$$

Substituting here the expressions for $g_{\eta\eta}$ and $\Phi_{\eta\eta}$, we obtain

$$\Delta\Phi = \Phi_{\xi\xi} + \Phi_{\eta\eta} = g_{\eta\eta} + 2\Phi_{\eta\eta} + \frac{Vb}{2a}\frac{1}{2\sqrt{\pi}}\exp\Big(-\frac{\eta^2}{4}\Big) = -\frac{Vb}{2a}\frac{1}{\sqrt{\pi}}\exp\Big(-\frac{\eta^2}{4}\Big). \tag{1.15}$$

To complete the formulation of the boundary-value problem for Eq. (1.12), we should impose the conditions on the solid boundary. According to the results obtained in [7, 8], we should not require satisfaction of the adhesion conditions on the entire solid boundary. (As mentioned above, this can lead to the divergence of the Dirichlet integral, i.e., to an infinite dissipation rate of the kinetic energy in the fluid and the loss of the physical sense of the solution.) Thus, the boundary $\eta = 0$ is divided into two parts: $0 < \xi < \xi_s$ and $\xi \geqslant \xi_s$, where $\xi_s$ is the small parameter. The adhesion conditions are imposed only on the second (larger) part of the boundary (they have already been used in deriving $\psi_m$ and $\tilde{\psi}_p$):

$$\Phi = 0, \quad \Phi_\eta = 0 \quad \text{for} \quad \eta = 0, \quad \xi \geqslant \xi_s. \tag{1.16}$$

The condition $\Phi = 0$ is preserved on the first part of the boundary, and the ideal slipping condition [7] is used instead of the second condition, which yields

$$\Phi_{\eta\eta} = -\frac{1}{\sqrt{\pi}}\frac{Vb}{2a}.$$

With account that $\Phi_{\xi\xi} = 0$, the conditions on the first part of the boundary are

$$\Phi = 0, \quad \Delta\Phi = -\frac{1}{\sqrt{\pi}}\frac{Vb}{2a} \quad \text{for} \quad \eta = 0, \quad 0 < \xi < \xi_s. \tag{1.17}$$

(The ideal slipping condition is chosen from various slipping conditions [7] to confine the hypothesis of local self-similarity of the flow near the moving contact line.) Thus, the mathematical problem (1.12), (1.13), (1.15)–(1.17) is posed. This problem is numerically solved below.

**2. Numerical Solution.** Since the flow region is not compact and there are singularities in the coefficients at $r = 0$, problem (1.12), (1.13), (1.15)–(1.17) is numerically solved in a quarter of a circle: $0 \leqslant \arctan(\eta/\xi) \leqslant \pi/2$ and $R_s^2 \leqslant \xi^2 + \eta^2 \leqslant R_b^2$, where $R_s$ and $R_b$ are chosen in a special manner (see below). For convenience of numerical solution in the chosen domain, the problem is rewritten in the polar coordinates $r_p = \sqrt{\xi^2 + \eta^2}$ and $\theta_p = \arctan(\eta/\xi)$, in which the chosen sector develops into a rectangle:

$$\Delta\Phi = -\omega, \tag{2.1}$$

$$\Delta\omega + \frac{1}{2}\Big(r_p\frac{\partial\omega}{\partial r_p} + \omega\Big) = 0, \tag{2.2}$$

where $\omega$ is the vorticity. The boundary conditions in the new terms are

$$\Phi = r_p - \frac{1}{\sqrt{\pi}}\int_0^{r_p} dl \int_0^{l} \exp\Big(-\frac{m^2}{4}\Big)dm, \quad \omega = \frac{1}{\sqrt{\pi}}\exp\Big(-\frac{r_p^2}{4}\Big) \tag{2.3}$$

for $\theta_p = \pi/2$ and

$$\Phi = 0, \qquad \Phi_{\theta_p} = 0 \quad \text{at} \quad r_p > r_s, \qquad \omega = 1/\sqrt{\pi} \quad \text{at} \quad r_p \leqslant r_s \tag{2.4}$$

for $\theta_p = 0$. (Since the factor $Vb/(2a)$ is present in all boundary conditions, it can be omitted for the present.)

The following conditions are imposed on specially introduced boundaries $r_p = R_s$ and $r_p = R_b$. For $r_p = R_s$, we seek the solution in the form of the Moffatt solution [9]: $\Phi = \sum_{k=1}^{\infty} r_p^k h_k(\theta_p)$. In the initial equation (2.2), because of the smallness of the second and third terms in its left side for $r_p \to 0$, we can ignore all the terms except for $\Delta\omega$. In fact, we seek a solution of the approximate equation $\Delta\Delta\Phi = 0$, where $\Delta = \partial^2/\partial r_p^2 + \partial/(r_p\partial r_p) + \partial^2/(r_p^2\partial\theta_p^2)$ is the Laplace operator in the polar coordinates. To resolve uniquely the resultant equation, we use the boundary conditions of continuity of $\Phi$ and $\omega$ at the interface points, i.e.,

$$\Phi\Big|_{\theta_p=\pi/2}(R_s) = \Phi\Big|_{r_p=R_s}\Big(\frac{\pi}{2}\Big), \qquad \Phi\Big|_{\theta_p=0}(R_s) = \Phi\Big|_{r_p=R_s}(0),$$

$$\omega\Big|_{\theta_p=\pi/2}(R_s) = \omega\Big|_{r_p=R_s}\Big(\frac{\pi}{2}\Big), \qquad \omega\Big|_{\theta_p=0}(R_s) = \omega\Big|_{r_p=R_s}(0).$$

This allows us to write the following condition to accuracy to small quantities of order $r_p^4$ for $r_p = R_s$:

$$\Phi = r_p^2\Big(\frac{1}{4\sqrt{\pi}}\cos 2\theta_p - \frac{1}{4\sqrt{\pi}}\Big), \quad \omega \equiv \frac{1}{\sqrt{\pi}}\Big(1 - \frac{r_p^2}{4}\Big). \tag{2.5}$$

The radius $R_b$ is chosen so that the conditions assumed valid for rather large $r_p$ were also valid, namely,

1) both velocity components $\partial\Phi/\partial\xi$ and $\partial\Phi/\partial\eta$ and vorticity $\omega$ tend to zero as $r_p \to \infty$;

2) after iqntroducing

$$\chi(r_p) = \Phi\Big|_{\theta_p=\pi/2} = r_p - (1/\sqrt{\pi})\int\limits_0^{r_p} dl \int\limits_0^{l} \exp(-m^2/4)\,dm$$

and differentiating

$$\partial\chi(r_p)/\partial r_p = 1 - (1/\sqrt{\pi})\int\limits_0^{r_p}\exp(-m^2/4)\,dm = 1 - (1/\sqrt{\pi})\Big(\int\limits_0^{\infty}\exp(-m^2/4)\,dm - \int\limits_{r_p}^{\infty}\exp(-m^2/4)\,dm\Big)$$

$$= (1/\sqrt{\pi})\int\limits_{r_p}^{\infty}\exp(-m^2/4)\,dm > 0$$

for all $r_p > 0$, we find that $\chi(r_p)$ is a monotonic nondecreasing function and $\chi(r_p) \approx c_3 > 0$ for $r_p \gg 1$. (It was found numerically that $c_3 \approx 1.13$.)

Thus, the following conditions should be valid for a properly chosen $R_b$:

$$\omega = 0, \quad \frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial r_p} = 0, \tag{2.6}$$

where $n$ is the external normal to the boundary, which should ensure compatibility at the interface points:

$$\omega\Big|_{\theta_p=\pi/2}(R_b) = \omega\Big|_{r_p=R_b}\Big(\frac{\pi}{2}\Big), \qquad \omega\Big|_{\theta_p=0}(R_b) = \omega\Big|_{r_p=R_b}(0),$$

$$\Phi\Big|_{\theta_p=\pi/2}(R_b) = c_3, \quad \Phi\Big|_{\theta_p=0}(R_b) = 0, \quad \frac{\partial\Phi}{\partial n}\Big|_{\theta_p=0}(R_b) = \frac{\partial\Phi}{\partial n}\Big|_{r_p=R_b}(0).$$

Thus, problem (2.1)–(2.6) is ready for numerical solution. It should be noted that the conditions for the stream function $\Phi$ are chosen at all four boundaries: the Dirichlet conditions on three boundaries and the Neumann condition on one boundary ($R_p = R_b$). For the vorticity $\omega$, the Dirichlet conditions are imposed on three boundaries. The Dirichlet condition is also satisfied on the fourth boundary ($\theta_p = 0$) for $r_p \leqslant r_s$, whereas the boundary condition for $r_p > r_s$ is obtained from the conditions for $\Phi$ in the course of numerical solution using the Thom formula [10].

Since of greatest interest is the region adjacent to the origin, we pass to the logarithmic coordinates $\gamma = \ln r_p$, and $\theta_p$ remains unchanged (which corresponds to the transition to a nonuniform mesh with smaller cells in the domain of interest). The equations take the form

$$\exp(-2\gamma)\Big(\frac{\partial^2\Phi}{\partial\gamma^2} + \frac{\partial^2\Phi}{\partial\theta_p^2}\Big) = -\omega; \tag{2.7}$$

$$\exp(-2\gamma)\Big(\frac{\partial^2\omega}{\partial\gamma^2} + \frac{\partial^2\omega}{\partial\theta_p^2}\Big) + \frac{1}{2}\Big(\frac{\partial\omega}{\partial\gamma} + \omega\Big) = 0, \tag{2.8}$$

and the boundary conditions are

$$\Phi\Big|_{\theta_p=\pi/2} = \exp(\gamma) - \frac{1}{\sqrt{\pi}}\int\limits_0^{\exp(\gamma)} dl \int\limits_0^{l}\exp\Big(-\frac{m^2}{4}\Big)dm, \quad \omega\Big|_{\theta_p=\pi/2} = \frac{1}{\sqrt{\pi}}\exp\Big(-\frac{\exp(2\gamma)}{4}\Big),$$

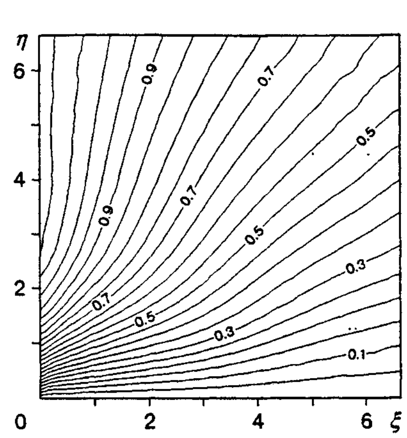


Fig. 1

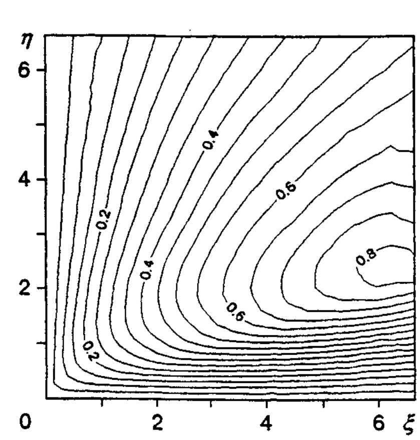


Fig. 2

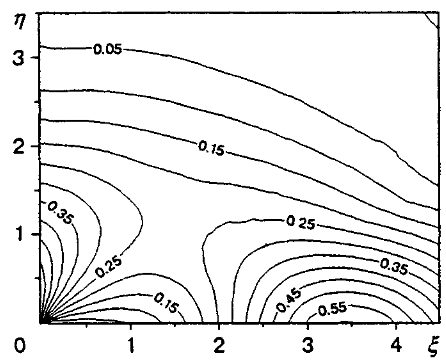


Fig. 3

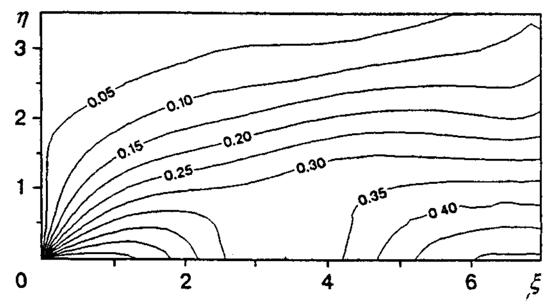


Fig. 4

$$\Phi\Big|_{\theta_p=0} = 0, \quad \gamma > \gamma_{sl}\colon \quad \Phi_{\theta_p}\Big|_{\theta_p=0} = 0, \qquad \gamma \leqslant \gamma_{sl}\colon \quad \omega\Big|_{\theta_p=0} = \frac{1}{\sqrt{\pi}}, \tag{2.9}$$

$$\Phi\Big|_{\gamma=\gamma_s} = \exp(2\gamma)\Big(\frac{1}{4\sqrt{\pi}}\cos 2\theta_p - \frac{1}{4\sqrt{\pi}}\Big), \quad \omega\Big|_{\gamma=\gamma_s} = \frac{1}{\sqrt{\pi}}, \quad \frac{\partial\Phi}{\partial\gamma}\Big|_{\gamma=\gamma_b} = 0, \quad \omega\Big|_{\gamma=\gamma_b} = 0.$$

Problem (2.7)–(2.9) is solved by the pseudo-transient method (i.e., a fictitious time is introduced) using the Peaceman–Rachford scheme by means of splitting equations relative to directions with a "cross" stencil. The difference equations for $\Phi$ and $\omega$ derived in this manner have the first order of approximation relative to the fictitious time and the second order relative to the spatial coordinates. The resultant scheme is absolutely stable. All four difference equations are solved by tridiagonal sweeping. The condition of diagonal prevalence is fulfilled for $\Phi$ automatically and for $\omega$ in the case of a rather small step in $\gamma$. To calculate the double integrals of probability (used in the condition on the free boundary and in deriving $\psi_m$ and $\tilde{\psi}_p$), we used the trapezoidal approximation.

**3. Numerical Results.** The results of numerical calculations are shown in Figs. 1–4 (they were obtained using a code written in PASCAL, and the calculated data arrays were graphically represented using the SULFER system of the Golden Software Company). Figure 1 shows the isolines of the calculated stream function $\Phi$ in the plane of the Cartesian coordinate system $(\xi, \eta)$ fitted with the moving contact line. Figure 2 shows the isolines of the total stream function (i.e., all three components $\Phi$, $\tilde{\psi}_p$, and $\psi_m$ are taken into account) on the plane $(\xi, \eta)$. The calculated vorticity $\omega = -\Delta\Phi(\xi, \eta)$ and the total vorticity (i.e., with account of the contribution of vorticities corresponding to $\tilde{\psi}_p$ and $\psi_m$) in the same coordinate system are plotted in Figs. 3 and 4, respectively.

The results show that the calculated vorticity $\omega$ never changes its sign and is nonnegative within the numerical domain considered. Based on these calculations, we propose the hypothesis that $\omega$ is nonnegative on the entire quarter of the plane $0 \leqslant r_p < \infty$, $0 \leqslant \theta_p \leqslant \pi/2$. To confirm the hypothesis proposed, we introduce a new function $u(r_p, \theta_p)$ such that $\omega(r_p, \theta_p) = \exp(-r_p^2/4)u(r_p, \theta_p)$. After that, the initial equations (2.1) and (2.2) become

$$\Delta\Phi = -\exp\Big(-\frac{r_p^2}{4}\Big)u(r_p, \theta_p), \tag{3.1}$$

$$\Delta u - \frac{1}{2}\Big(r_p\frac{\partial u}{\partial r_p} + u\Big) = 0. \tag{3.2}$$

Equation (3.2) is elliptical, like Eq. (2.2); contrary to the latter, it satisfies the principle of the maximum [11]. The boundary conditions (2.3) and (2.4) on the solid and free boundaries for $\omega(r_p, \theta_p)$ yield the following conditions for $u(r_p, \theta_p)$:

$$u = \frac{1}{\sqrt{\pi}} \quad \text{for} \quad \theta_p = \frac{\pi}{2}, \qquad u = \frac{1}{\sqrt{\pi}}\exp\Big(\frac{r_p^2}{4}\Big) \quad \text{for} \quad \theta_p = 0, \quad r_p \leqslant r_s$$

(if $r_p > r_s$, the condition for $u$ is found from the conditions for $\Phi$ in the course of numerical solution using the Thom formula; the values of $u$ for $r_p > r_s$ are positive). The assumption that $u$ is nonnegative and $\omega$ is limited allows us to conclude that the vorticity $\omega$ exponentially decreases as $r_p \to \infty$.

From the physical viewpoint, the proposed hypothesis has the following meaning. In the Cartesian coordinate system $(x, y)$ introduced previously and corresponding to $(r_p, \theta_p)$, the shear stress $P_{x,y}$ related to the stream function $\Phi$ can be rewritten as follows [6]:

$$P_{x,y} = \rho\nu\Big(\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2}\Big) = \rho\nu\Big(\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2}\Big) = -\rho\nu\omega$$

for $y = 0$, since $\partial^2\Phi/\partial x^2 = 0$ because of the condition $\Phi = 0$ on the wall. Thus, the unchanged sign of $\omega$ is identical to the unchanged sign of the corresponding shear stress $P_{xy}$. If $\omega \geqslant 0$ within the entire region under study in accordance with the hypothesis, we have $P_{xy} \leqslant 0$, which agrees with the physics of the process.

Finally, it should be noted that the qualitative results are not changed by varying the small parameter of the problem $r_s$ within 0.0007 to 0.02.

**4. Comments.** First of all, we have to estimate the dimensionless parameters for a typical situation. If we choose water at room temperature with $a = b = 10$ cm and $V = 10$ cm/sec as a fluid, we obtain $\mathrm{Re} = 10^4$, $\mathrm{Ca} = 1.33 \cdot 10^{-3}$, and the Mach number $\mathrm{M} = V/\tilde{c} = 6.7 \cdot 10^{-3}$ ($\tilde{c}$ is the speed of sound). This means that the influence of fluid compressibility is negligibly small for times of the order of $10^{-4}$ sec or greater. In other words, the effect of fluid compressibility is localized in time, whereas the effect of viscosity is localized in space. For the time moment $t = 10^{-4}$, the thickness of an unsteady boundary layer near the solid boundaries is $\delta = \sqrt{\nu t} = 10^{-3}$ cm, i.e., slightly smaller than the initial height of the fluid column.

Up to now, the role of gravitation was not taken into account in the process considered. It is important for high Bond numbers $\mathrm{B} = \rho\tilde{g}a^2/\sigma$, where $\tilde{g}$ is the acceleration of gravity. In particular, we cannot ignore the effect of gravitation for a given set of parameters under ground-based conditions, though it is possible to conduct the corresponding experiment under the conditions of a practically zero-gravity state. Another possibility is to decrease the linear scale and simultaneously increase the characteristic velocity. For example, if $a = 0.1$ cm and $V = 30$ cm/sec, we obtain $\mathrm{Re} = 300$, $\mathrm{Ca} = 0.04$, and $\mathrm{B} = 0.13$ for the ground-based conditions.

The next important issue is the physical realization of the Ovsyannikov solution. The initial data for this solution do not agree with the velocity distribution resulting from the instantaneous motion of the solid plates toward each other. (This distribution for a planar analog of the problem was obtained in [12], and the transitional process with account of fluid compressibility for small times was studied in [13].) As noted in [12], if the ratio $a/b$ is small, then the linear distribution of velocity in the initial data is close to that predicted by the classical theory of hydrodynamic hammer. The attempt to realize the same initial conditions for greater values of $a/b$ leads to a quadratic dependence of the pulse distribution of the initial pressure at the side boundary versus the vertical component (which can, probably, be ensured by explosive loading of the fluid).

A critical moment of the model proposed is the assumption that the contact angle equals $\pi/2$. Otherwise, we would have neither the exact solutions, which describe the motion of an ideal capillary fluid, nor any results on solvability of the corresponding initial-boundary problem for the Euler equations. However, if this solution is known (for instance, numerically), it is possible to construct an unsteady axisymmetric boundary layer near the free boundary of the fluid bridge following the method proposed by Batishchev and Srubshchik [14] who considered a similar planar problem.

The specific role of $\pi/2$ is related to the fact that, for an arbitrary value of the contact angle, the solution of the corresponding problem for the Navier–Stokes equations has, strictly speaking, a power singularity at the corner point of the boundary [15]. This circumstance requires a more detailed study of the flow structure inside the corner region. The free part of the boundary is no longer asymptotically linear; probably, this line will have an infinite curvature at the contact point in the plane $(r, z)$.

Finally, we should briefly discuss the choice of the slipping condition. The hypothesis of local self-similarity of the flow in the corner region simplifies our consideration and at the same time corresponds to the self-similar character of the boundary-layer flow. Still, it is desirable to have some additional arguments for a rational choice of the small parameter $r_s$. It should be taken into account that, for an arbitrary value of the parameter $r_s$, the shear stress on the solid wall in the solution of the problem posed increases infinitely at the point where the adhesion condition is replaced by the ideal slipping condition (this fact follows from the results of [8]). At the same time, according to [16], the proper choice of this parameter allows one to set contraints on the shear stress.

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